The path-distance-width of hypercubes

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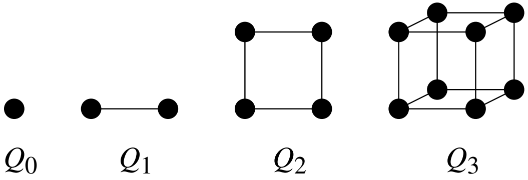
Hypercube

Definition (Hyercube)

The *d-dimensional hypercube* Q_d, or d-cube, is the graph with

- $V(Q_d) =$ the set of all binary strings of length d,
- $E(Q_d) =$ the pairs of strings of Hamming distance 1.



Properties of hypercubes

- $|V(Q_d)| = 2^d$, $|E(Q_d)| = d \cdot 2^{d-1}$ $(\because Q_d$ is d-regular)
- diameter of Q_d is d (attained by 0^d and 1^d)

We study a width parameter of hypercubes.

Width parameters of graphs

Several width parameters are studied in algorithmic graph theory

Parameters from industrial applications

- Band-width: minimizing maximum dilation in a linear circuit
- Cut-width: minimizing maximum congestion in a linear circuit

Graph minor related parameters

- Tree-width: measures how close a graph is to a tree
- Path-width: measures how close a graph is to a path
- Branch-width, Carving-width, etc.

Many intractable graph problems become easy for graphs of bounded width parameters.

Our width parameter

- Path-distance-width: minimizing the width of BFS

Width parameters of hypercubes

Width parameters of hypercubes

Studies on width parameters of hypercubes give better understanding on these parameters. This is because:

- Usually, hypercubes have large width parameters;
- But basic tools (for l.b.) do not work for hypercubes (eg. degree);
- Thus wee need a nice tool for a good lower bound;

Known results

- Cut-width (e) by [Harper (1964)]
- Band-width (e) by [Harper (1966)]
- Path-width (e) and Treewidth (a) by [Chandran & Kavitha (2006)]
- Carving-width (e) by [Chandran & Kavitha (2006)]

e: exact, a: asymptotic

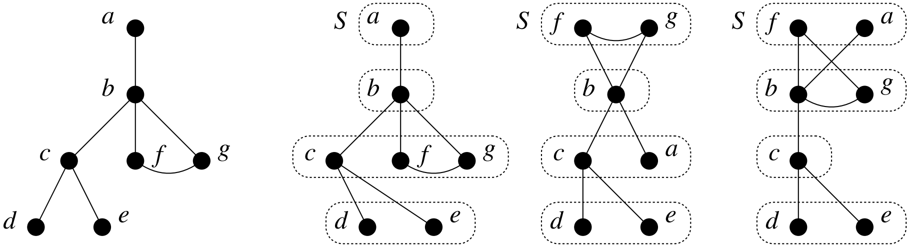
Definition: Path-distance-width (1 of 2)

Definition (Distance structure)

$D(S) = (L_0, \ldots, L_t)$ is a distance structure of G rooted at S if

- $\bigcup_{0 \le i \le t} L_i = V(G)$, and
- $L_i = \{v \in V(G) \mid d(S, v) = i\}$,

where $d(S, v) = \min_{u \in S} d(u, v)$.



Definition: Path-distance-width (2 of 2)

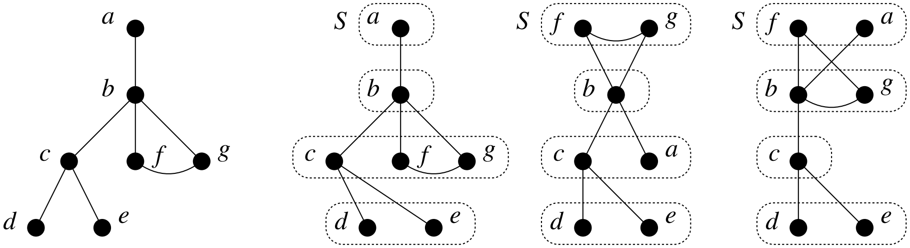
Definition (Path-distance-width)

The *path-distance-width* of G with initial set S, denoted $\mathrm{pdw}_S(G)$, is

$$\mathrm{pdw}_S(G) = \max_{L_i \in D_G(S)} |L_i|.$$

The *path-distance-width* of G is defined as

$$\mathrm{pdw}(G) = \min_{S \subseteq V(G)} \mathrm{pdw}_S(G).$$



∗ pdw is defined for connected graphs only.

History of path-distance-width

Some algorithmic results are known.

- Introduced to study the graph isomorphism problem [Yamazaki, Bodlaender, de Fluiter, Thilikos (1997)]
  - Determining the pdw of a graph is NP-hard
  - For pdw bounded graphs, GI can be solved in poly time.
- Approximation hardness for trees [Yamazaki (2001)]
- Constant-factor approximation algorithms for graphs with path-like structures [O. *et al.* (2011)]
- An improved algorithm for the graph isomorphism problem [O. (2012)]
  - For a subclass of the class of pdw bounded graphs, GI can be solved in FPT time.

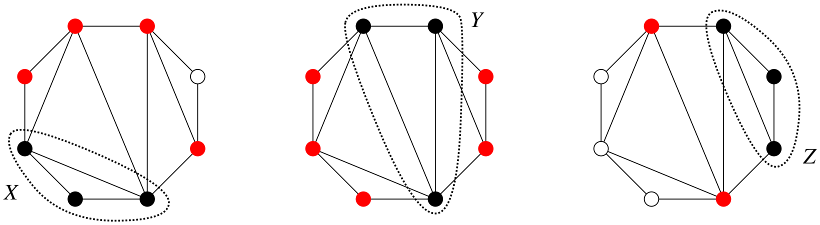
Vertex- and edge-boundaries

Definition (Vertex-boundary)

Let G be a graph, $X \subseteq V(G)$, and $s \in \{1, \ldots, |V(G)|\}$.

- Vertex boundary: $\partial_v(X) = \{v \in V(G) \setminus X \mid \exists u \in X, \{u, v\} \in E(G)\}$
- Vertex isoperimetric value: $\partial_v(s) = \min_{|X|=s} |\partial_v(X)|$

$*$ Edge boundry ∂_e is defined analogously.



$|\partial_v(X)| = 4$ $|\partial_v(Y)| = 5$ $|\partial_v(Z)| = 2$

In this example, $\partial_v(3) = 2$.

Isoperimetric value based lower bounds

The following lower bounds are developed to determine the width parameters of hypercubes, but hold for any graphs.

Theorem (Chandran & Subramanian (2005))

tree-width$(G) \geq \min_{s/2 \leq i \leq s} \partial_v(i)$ *for any* $s \leq |V(G)|$.

Theorem (Chandran & Kavitha (2006))

carving-width$(G) \geq \min_{s/2 \leq i \leq s} \partial_e(i)$ *for any* $s \leq |V(G)|$.

We present a lower bound in a similar form:

Theorem

For any w *and* s *with* $w \leq s \leq |V(G)|$,

$$\mathrm{pdw}(G) \geq \min\left\{w, \min_{s-w \leq i \leq s} \partial_v(i)\right\}.$$

Our reulst

We present a general lower bound on path-distance-width of graphs, and determine the path-distance-width of hypcubes by applying the bound.

Theorem (General lower bound)

For any w and s with $w \leq s \leq |V(G)|$,

$$\mathrm{pdw}(G) \geq \min\left\{w, \min_{s-w \leq i \leq s} \partial_v(i)\right\}.$$

Theorem (The path-distance-width of hypercubes)

For any d,

$$\mathrm{pdw}(Q_d) = \binom{d}{\lfloor d/2 \rfloor}.$$

Upper bound

The upper bound can be achieved by taking only one vertex.

Lemma

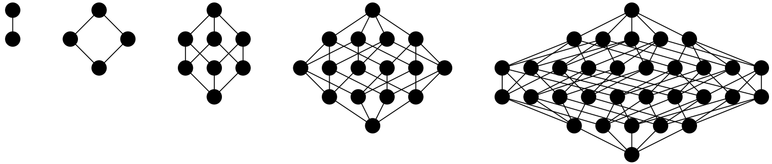
$$\mathrm{pdw}(Q_d) \le \binom{d}{\lfloor d/2 \rfloor}.$$

Proof.

Let $D(\{0^d\}) = (L_0, \ldots, L_t)$ is the distance structure rooted at $\{0^d\}$. Then

$$L_i = \{u \in Q_d \mid i \text{ has exactly } i \text{ non-zero entries}\}.$$

Thus $\max_i |L_i| = \max_i \binom{d}{i} = \binom{d}{\lfloor d/2 \rfloor}$. □



$\mathrm{pdw}(Q_4) \le \binom{4}{\lfloor 4/2 \rfloor} = 6$ $\quad$ $\mathrm{pdw}(Q_5) \le \binom{5}{\lfloor 5/2 \rfloor} = 10$

Naïve lower bounds

We have two general lower bounds. They are too weak for hypecubes.

Lemma

For any connected graph G with minimum degree $\delta(G)$*,*

$$\mathrm{pdw}(G) \geq (\delta(G) + 1)/2.$$

$\implies \mathrm{pdw}(Q_d) \geq (d + 1)/2.$

Lemma

For any connected graph G with diameter $\mathrm{diam}(G)$*,*

$$\mathrm{pdw}(G) \geq |V(G)|/(\mathrm{diam}(G) + 1).$$

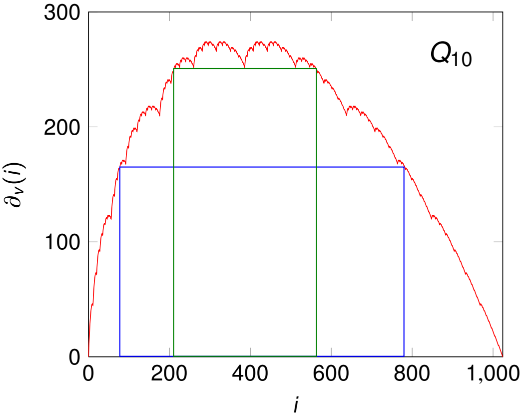
$\implies \mathrm{pdw}(Q_d) \geq 2^d/(d + 1)$. $\binom{d}{d/2} \sim 2^d/\sqrt{\pi d/2}$

Lower bound

Theorem

For any w and s with $w \leq s \leq |V(G)|$,

$$\textit{path-distance-width}(G) \geq \min\left\{w, \min_{s-w \leq i \leq s} \partial_v(i)\right\}.$$



Finding a fat rectangle below the red line.

- width $= w$
- height $= \min_{s-w \leq i \leq s} \partial_v(i)$

min{width, height} gives a l.b.

Blue rectangle: 700×160

Green rectangle: 350×250

Proof of the lower bound

Remark

Let $(L_0, \ldots, L_t)$ be a distance structure of G. Then, for any $j \le t - 1$,

$$\partial(\bigcup_{i=0}^{j} L_i) = L_{j+1}.$$

Theorem

For any w and s with $w \le s \le |V(G)|$,

$$\textit{path-distance-width}(G) \ge \min\left\{w, \min_{s-w \le i \le s} \partial_v(i)\right\}.$$

Proof.

Let $(L_0, \ldots, L_t)$ be a distance structure of G.

- $|\bigcup_{i=0}^{j} L_i| \in \{s - w, \ldots, s\}$ for some $j \implies |L_{j+1}| \ge \min_{s-w \le i \le s} \partial_v(i)$.
- Otherwise, $\exists j$ s.t. $|\bigcup_{i=0}^{j} L_i| < s - w$ and $|\bigcup_{i=0}^{j} L_i \cup L_{j+1}| > s$.

□

Isoperimetric ordering for hypercubes

For a binary string u, let $\mathrm{pop}(u)$ denote the number of 1's in u.

Definition (Simplicial ordering)

The *simplicial ordering* $\prec$ on the binary strings of length d is an odering such that $u_1 u_2 \dots u_d \prec v_1 v_2 \dots v_d \iff \mathrm{pop}(u) < \mathrm{pop}(v)$, or $\mathrm{pop}(u) = \mathrm{pop}(v)$ and there exists j s.t. $u_j > v_j$ and $u_i = v_i$ for $i < j$.

eg: $000 \prec 100 \prec 010 \prec 001 \prec 110 \prec 101 \prec 011 \prec 111$

Theorem (Harper 1966)

Let X_s be the set of the first s vertices of Q_d in the simplicial order. Then

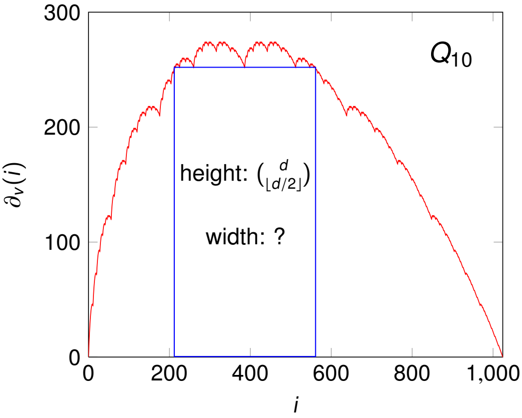
$$\partial_v(s) = |\partial_v(X_s)|.$$

The above theorem is developed to determine the band-width of Q_d

Theorem (Harper 1966 and Wang, Wu, Dumitrescu 2009)

$$\textit{band-width}(Q_d) = \sum_{i=0}^{d-1} \binom{i}{\lfloor i/2 \rfloor}.$$

Ploting isoperimetric values of Q_d



We know: $\mathrm{pdw}(Q_d) \leq \binom{d}{\lfloor d/2 \rfloor}$

Task: finding the widest rectangle with height $\binom{d}{\lfloor d/2 \rfloor}$ below the red line.

Proof of the lower bound

Let $s = \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{i}{\lfloor i/2 \rfloor}$ and $w = \binom{d}{\lfloor d/2 \rfloor}$. We can show that, for $s - w \leq i \leq s$, $\partial_V(i) \geq \binom{d}{\lfloor d/2 \rfloor}$ by exploiting the structure of the simplicial ordering.

There is a shortcut. Let $f(m) = \sum_{i=1}^{m/2} \binom{2i-1}{i}$ for even m.

Theorem (Kleitman 1986)

If d is even and $s \in \{2^{d-1} - \frac{1}{2}\binom{d}{d/2} - f(d) + 1, \ldots, 2^{d-1} + f(d-2)\}$ or d is odd and $s \in \{2^{d-1} - f(d+1) + 1, \ldots, 2^{d-1} + f(d-1)\}$, then

$$\partial_V(s) \geq \binom{d}{d/2}.$$

It suffices to show that these ranges are wide enough
$\Longleftarrow$ just a routine task.

Conclusion

Theorem (General lower bound)

For any w and s with $w \leq s \leq |V(G)|$, $\mathrm{pdw}(G) \geq \min\left\{w, \min_{s-w \leq i \leq s} \partial_V(i)\right\}$.

Theorem (The path-distance-width of hypercubes)

For any d, $\mathrm{pdw}(Q_d) = \binom{d}{\lfloor d/2 \rfloor}$.

More applications of the lower bound?

- For grids (Cartesian products of paths), the generalized simplicial ordering gives isoperimetric values.
- For even tori (Cartesian products of cycles of even length), the generalized simplicial ordering gives isoperimetric values.
- For Hamming graphs (Cartesian products of complete graphs), no such ordering is known.

Other (more applicable) lower bounds?

Thank you!