Clique minors, chromatic numbers for degree sequence

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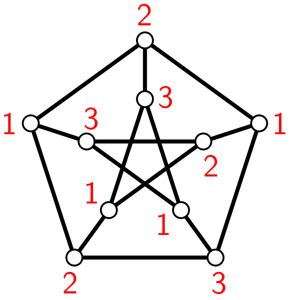
2013.1.26 Hakata

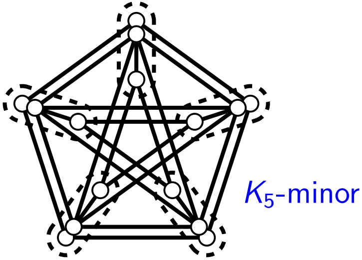
Chromatic number and Hadwiger number

Let G be a graph. (We only consider *simple* graphs.)

- $\chi(G)$: the chromatic number of G.
- $h(G)$: maximum size of clique minors in G, called the Hadwiger number of G.



$\chi(G) = 3$



$h(G) = 5$

Hadwiger's conjecture

Hadwiger's Conjecture (1943)

Every graph with chromatic number k has a K_k-minor.
(Equivalently, $\forall G,\ h(G) \geq \chi(G)$.)

Hadwiger's conjecture

- was proved for $k = 4$ by Dirac (1952);
- for $k = 5$ implies Four Color Theorem (FCT);
- is affirmative for $k = 5$ by FCT and Wagner (1937);
- is affirmative for $k = 6$ by Robertson, Seymour and Thomas (1993) using FCT;
- is open for $k \geq 7$.

Hadwiger's conjecture for degree sequences

Let $D = (d_1, d_2, \ldots, d_n)$ be a degree sequence of a graph.

- $\chi(D) := \max\{\chi(G) \,:\, G$ has deg. seq. $D\}$.
- $h(D) := \max\{h(G) \,:\, G$ has deg. seq. $D\}$.

Robertson and Song (2009) posed:

Hadwiger's Conjecture for Degree Sequences

For every degree sequence D, $h(D) \geq \chi(D)$ holds.

- If Hadwiger's conjecture is true, then Hadwiger's conjecture for degree sequences is also true.

Hadwiger's conjecture for degree sequences

Hadwiger's Conjecture for Degree Sequences

For every degree sequence D, $h(D) \geq \chi(D)$ holds.

Theorem (Robertson, Song 2009)

Hadwiger's conjecture for degree sequences is true for all near regular degree sequences.

A degree sequence $D = (d_1, d_2, \ldots, d_n)$ is said to be near regular if $\max_i\{d_i\} - \min_i\{d_i\} \leq 1$.

Dvořák and Mohar have proved!

Recently, Hadwiger's Conjecture for Degree Sequences was confirmed by showing a stronger statement.

Theorem (Dvořák, Mohar 2012+)

For every degree sequence D, $h'(D) \geq \chi(D)$ holds.

- $h'(D) := \max\{h'(G) \ : \ G$ has deg. seq. $D\}$.
- $h'(G)$: maximum k such that G has a *topological K_k-minor*.
- A topological K_k-minor of a graph is a subgraph isomorphic to a subdivision of K_k.
- Note: $h(G) \geq h'(G)$, and hence $h(D) \geq h'(D)$.

Note on $h'(G)$: Hajós' number

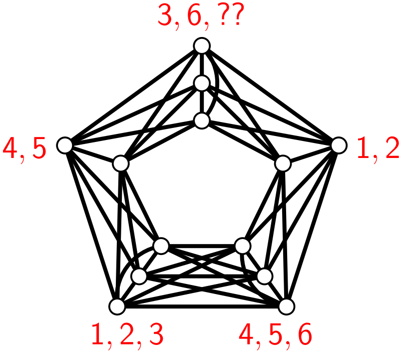
(known as) Hajós' Conjecture

$\forall G,\ h'(G) \geq \chi(G)$.
(Every graph with $\chi = k$ has a topological K_k-minor.)

Hajós' conjecture

- implies Hadwiger's conjecture, since $h(G) \geq h'(G)$;
- is true for $k \leq 4$ by Dirac (1952);
- for $k = 5$ implies Four Color Theorem (FCT);
- is false for $k \geq 7$ by Catlin (1979);
- is false for almost all graphs, by Erdős and Fajtlowicz (1981);
- is open for $k = 5, 6$.

Counterexample to Hajós conjecture



$$\chi(G) = 7 \ > \ h'(G) = 6.$$

Hajós' conjecture

(known as) Hajós' Conjecture

$\forall G,\ h'(G) \geq \chi(G)$.
(Every graph with $\chi = k$ has a topological K_k-minor.)

Hajós' conjecture

- implies Hadwiger's conjecture, since $h(G) \geq h'(G)$;
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Dvořák and Mohar's Result

Theorem (Dvořák, Mohar 2013+)

For every degree sequence D, $h'(D) \geq \chi(D)$ holds.

- Their proof involves a lot, and is complicated.
- They did not determine the exact values of $h'(D)$ or $\chi(D)$.

We shall give

1. an alternative and very short proof of $h(D) \geq \chi(D)$; (Unfortunately, our argument does not work for proving $h'(D) \geq \chi(D)$ so far.)
2. the exact values of $h'(D)$ for near regular case;
3. a good bound for $\chi(D)$ for (near) regular case;

Some Remarks on Hajós' Numbers and Chromatic Numbers for Degree Sequences

Observations

Suppose $D = (d_1, d_2, \ldots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n$.

- If $h'(D) = k$, then we have $d_k \geq k - 1$. This means:

$$h'(D) \ \leq \ \max\{k \mid d_k \geq k - 1\}.$$

  - Note that, $h(D)$ can be as large as $\sqrt{n}$ even when $d_1 = \cdots = d_n = 3$.

- We can greedily color the graph with the degree sequence D using at most $\max\{k \mid d_k \geq k - 1\}$ colors.
- So if the equality $h'(D) = \max\{k \mid d_k \geq k - 1\}$ holds, then we conclude $h'(D) \geq \chi(D)$ as required.
- However, this is *not true* in general.

Results for regular degree sequences

- $D = (d, d, \dots, d) = (d^n)$, $(0 \le d \le n-1,\ dn:$ even$)$.
- $\overline{d} := n - 1 - d$.

Theorem 1

$$h'(D) = \begin{cases} d+1 & \text{if } d \le (n-1)/2; \\ \left\lfloor \left(\frac{1}{2} + \frac{1}{2\overline{d}+2}\right) n \right\rfloor & \text{if } d > (n-1)/2. \end{cases}$$

Theorem 2

$$\chi(D) \le \begin{cases} d+1 & (\text{if } d \le (n-1)/2); \\ \left\lfloor \left(\frac{1}{2} + \frac{1}{4\overline{d}+2}\right) n \right\rfloor & \text{if } d > (n-1)/2. \end{cases}$$

Proof (the upper bound for $h'(D)$)

Show that: $h'(D) \leq \left(\frac{1}{2} + \frac{1}{2\overline{d}+2}\right)n$.

- Let G be a d-regular n-vertex graph with $h'(G) = k$.
- Let X be the set of branch vertices of a top. K_k-minor. $Y := V(G) - X$.
- Let r be the number of nonadjacent pairs in X.
- $e_{\overline{G}}(X, Y) = \sum_{x \in X} d_{\overline{G}}(x) - 2r = \overline{d}|X| - 2r = \overline{d}k - 2r$.
- $e_{\overline{G}}(X, Y) \leq \sum_{y \in Y} d_{\overline{G}}(y) = \overline{d}|Y| = \overline{d}(n-k)$.
- There are at least r subdividing vertices in Y, hence $r \leq |Y| = n - k$.

$$\overline{d}(n-k) \geq \overline{d}k - 2r \geq \overline{d}k - 2(n-k),$$

$$(2\overline{d}+2)k \leq (\overline{d}+2)n. \qquad \square$$

Exact value of $h'(D)$ for near regular case

- $D = ((d+1)^p, d^{n-p})$
  $(0 \le d \le n-1,\ 0 \le p \le n-1,\ dn+p : \text{even})$.
- $\overline{d} := n - 1 - d$.

Theorem 3

$$h'(D) = \begin{cases} d+2 & \text{if } d \le \frac{n-2}{2} \text{ and } p \ge d+2; \\ d+1 & \text{if } d \le \frac{n-2}{2} \text{ and } p \le d+1; \\ \left\lfloor \frac{(\overline{d}+2)n+p}{2\overline{d}+2} \right\rfloor & \text{if } d \ge \frac{n-1}{2} \text{ and } p \le \frac{(\overline{d}+2)n}{2\overline{d}+1}; \\ \left\lfloor \frac{(\overline{d}+2)n-p}{2\overline{d}} \right\rfloor & \text{if } d \ge \frac{n-1}{2} \text{ and } p > \frac{(\overline{d}+2)n}{2\overline{d}+1}. \end{cases}$$

A Short Proof of
Hadwiger's Conjecture for Degree Sequences

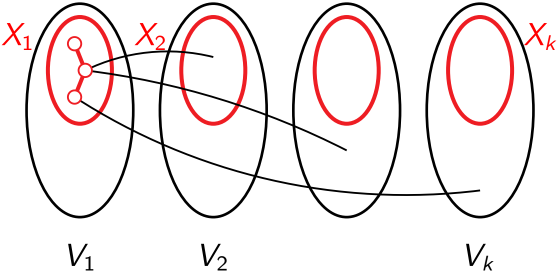
Definition: k-CDP

- Let $(V_1, V_2, \dots, V_k)$ be a partition of $V(G)$.
- $(V_1, V_2, \dots, V_k)$ is said to be a connected dominating partition of size k (k-CDP for short)

  if for $1 \le \forall i \le k$, $\exists X_i$: a connected component of $G[V_i]$ such that $E(X_i, V_j) \neq \emptyset$ for every $j \neq i$.
- The CDP number of G:

$$\rho(G) := \max\{k \mid G \text{ has a } k\text{-CDP}\}.$$

Definition: k-CDP



Observations on CDP number

Proposition 1

$\forall G,\ \chi(G) \leq \rho(G)$.

Proof:

- Let $k = \chi(G)$, and let $V_1, \ldots, V_k$ be the color classes.
- Then, for each i, $\exists x_i \in V_i$ s.t. $E(x_i, V_j) \neq \emptyset$ for $\forall j$, for otherwise we can recolor all vertices of V_i without using color i.
- Put $X_i = \{x_i\}$, then we obtain a k-CDP $(V_1, \ldots, V_k)$. $\square$

Observations on CDP number

Proposition 1

$\forall G,\ \chi(G) \le \rho(G)$.

Proposition 2

$\forall G,\ h(G) \le \rho(G)$.

Proof:

- Let $k = h(G)$, and let $X_1, \ldots, X_k$ be disjoint sets of vertices such that the contraction of X_i into v_i $(1 \le i \le k)$ yields a complete graph on $\{v_1, \ldots, v_k\}$.
- Expand each X_i into V_i to obtain a partition $(V_1, \ldots, V_k)$ of $V(G)$, which is a k-CDP of G. □

Observations on CDP number

Proposition 1

$\forall G,\ \chi(G) \leq \rho(G).$

Proposition 2

$\forall G,\ h(G) \leq \rho(G).$

- $\rho(D) := \max\{\rho(G) : G \text{ has deg. seq. } D\}.$

Corollary

$\forall D,\ \chi(D) \leq \rho(D)$ and $h(D) \leq \rho(D).$

Main Theorem

Theorem 4

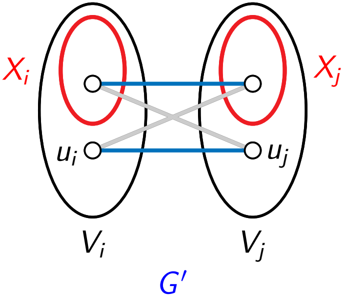
$\forall D$, $h(D) = \rho(D)$. Consequently, $\chi(D) \leq h(D)$.

Proof (1/2)

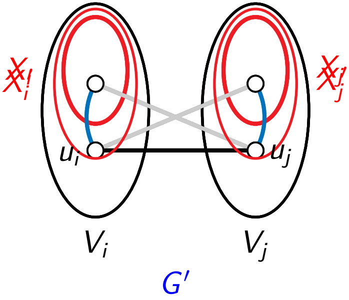
- We need to prove that $h(D) \geq \rho(D)$.
- Let $k = \rho(D) = \rho(G)$.
- Let $(V_1, \ldots, V_k)$ be a k-CDP, with a conn. cpt. X_i in V_i.
- If $E(X_i, X_j) \neq \emptyset$ for all pairs i, j, then by contracting each X_i into a single vertex, we obtain a K_k. Thus,
$$h(D) \geq h(G) \geq k = \rho(D).$$
- Otherwise, $E(X_i, X_j) = \emptyset$ for some i, j.

Proof (2/2)

Case 1: $u_i u_j \notin E(G)$



Case 2: $u_i u_j \in E(G)$



Open Problems

- Give a short proof of $h'(D) \geq \chi(D)$.
- Determine $h'(D)$ for all degree sequences D, or give an algorithm determining $h'(D)$ for given D.
- Give a better upper bound for $\chi(D)$ for (near) regular degree sequences D.
  Our bound $\chi(D) \leq \left\lfloor \left(\frac{1}{2} + \frac{1}{4\bar{d}+2}\right) n \right\rfloor$ for regular degree sequences is sharp for $d \in \{n-1, n-3, n/2\}$.
- Consider $\min\{h(G)\}$, $\min\{h'(G)\}$ and $\min\{\chi(G)\}$ of the graphs with a given degree sequence.