The Algebraic Connectivity of Graphs

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Outline

- Introduction
- Two main techniques for algebraic connectivity
- Bounds for the algebraic connectivity
- Extremal graphs with maximum (minimum) algebraic connectivity
- The algebraic connectivity of random graphs

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Definition and Notation

- $G = (V(G),\ E(G))$ a simple graph,
  vertex set $V(G) = \{v_1, \cdots, v_n\}$
  edge set $E(G) = \{e_1, \cdots, e_m\}$.
- $D(G) = diag(d_1, \cdots, d_n)$: degree diagonal matrix
  d_i : degree of vertex v_i (the number of edges incident to v_i).
- There are several matrices associated with a graph
- $A(G) = (a_{ij})$: Adjacency matrix of G,
  $a_{ij} = 1$ if $v_i \sim v_j$ and $a_{ij} = 0$ otherwise.
- $A(G)$ is a nonnegative symmetric $(0, 1)$ matrix with the zeros on the main diagonal.

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Definition and Notation

- Laplacian Matrix of a graph

$$L(G) = D(G) - A(G)$$

- The smallest eigenvalue of $L(G)$ is 0
- Fiedler (1973) The second smallest eigenvalue $\alpha(G)$ of $L(G)$ is called the algebraic connectivity of G.

Theorem 1

(Fiedler 1973) $\alpha(G) > 0$ if and only if G is connected.

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Background

Theorem 2

(Fiedler 1973) Let G be a graph with vertex connectivity $\nu(G)$, edge connectivity $\nu'(G)$ and the minimum degree $\delta(G)$. Then

$$\alpha(G) \leq \nu(G) \leq \nu'(G) \leq \delta(G).$$

- $\alpha(G)$ serves as a measure of connectivity of a graph.

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Background

- On combinatorial optimization problems: the problem of certain flowing process, the maximum cut problem and the traveling salesman problem.
- Fiedler vectors are used in algorithms for distributed memory parallel processors.
- The algebraic connectivity is a measure of the robustness in complex networks.
- Application to Continuous or Digital Space.
- Algebraic connectivity may explain the evolution of gene regulatory networks.

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Basic properties

- $$\alpha(G) = \min_{f \perp e, f \neq 0} \frac{< L(G)f, f >}{< f, f >},$$
  $$= \min_{f \neq 0, \sum_{u \in V} f(u) = 0} \frac{\sum_{uv \in E(G)} (f(u) - f(v))^2}{\sum_{u \in V} f(u)^2}$$
  where e is all one vector.
- If G is a graph of order n. Then
  $$\alpha(G) = n - \lambda_{max}(\overline{G}),$$
  where $\overline{G}$ is the complement of G.
- $\alpha(G + e) \geq \alpha(G)$.

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Basic properties

- $\alpha(K_n) = n.$
- $\alpha(P_n) = 2(1 - \cos\frac{\pi}{n}).$
- $\alpha(C_n) = 2(1 - \cos\frac{2\pi}{n}).$
- $\alpha(K_{p,q}) = \min\{p, q\}.$
- $\alpha(Persen\ graph) = 2.$

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Two Main techniques

- : 1. Nonnegative matrix theory
- For a tree T with a vertex v, a branch of T at v is one of the connected components of T which results from removing vertex v and all edges incident with it.
- Bottleneck matrix of a branch at vertex k: the diagonal block of L_k^{-1}, where L_k is the principal submatrix of $L(G)$ by deleting the $k-$th row and column of $L(G)$.
- Perron value of a branch of T at vertex k is the Perron value (the spectral radius) of the corresponding bottleneck matrix.
- A branch at k is a Perron branch if the Perron value of that branch is the same as the spectral radius of L_k^{-1}.

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1.Nonnegative matrix theory

Theorem 3

(Kirkland, Neumann and Shader 1996) Let T *be a tree on* n *verteices*$\{1, \cdots, n\}$ *If* $i \sim j$*, then* T *is type* II *(no component of an eigenvector of* $L(T)$ *corresponding to* $\alpha(G)$ *is 0) if and only if there exists a* $0 < \varepsilon < 1$ *such that* $\rho(M_1 - \varepsilon J) = \rho(M_2 - (1-\varepsilon)J)$*, where* M_1 *is the bottleneck matrix for the branch at* j *containing* i*, and* M_2 *is the bottleneck matrix for the branch at* i *containing* j*. Moreover,*

$$\alpha(T) = \frac{1}{\rho(M_1 - \varepsilon J)} = \frac{1}{\rho(M_2 - (1-\varepsilon)J)}.$$

1.Nonnegative matrix theory

Theorem 4

(Kirkland, Neumann and Shader 1996) Let T be a tree on n verteices$\{1, \cdots, n\}$ If $i \sim j$, then T is type I (the k-th component of an eigenvector of $L(T)$ corresponding to $\alpha(G)$ is 0) if and only if there exist two or more Perror branch of T at k. Moreover,

$$\alpha(T) = \frac{1}{\rho(L_k^{-1})}.$$

1. Nonnegative matrix theory

Theorem 5

(Kirkland and Neumann 1997) Let T be a tree on n vertices $\{1, \cdots, n\}$ and M be the bottleneck matrix of a branch B of T at k, which does not contain all of the characteristic vertices of T. Let $\widetilde{T}$ be a tree from T by replacing the branch at k by some other branch $\widetilde{B}$ at k whose bottleneck matrix is $\widetilde{M}$. If $M \ll \widetilde{M}$ (there exist two permutation matrices P, Q such that PMP^T is entrywise dominated by a principal submatrix of $Q\widetilde{M}Q^T$), then $\alpha(\widetilde{T}) \leq \alpha(T)$.

2. Graph Transformation

2. Use Graph perturbation.

Theorem 6

(Guo 2010) Let G_1 and G_2 be two graphs with at least two vertices, respectively. If G' is a graph by joining an edge from a vertex u of G_1 and a vertex v of G_2, and G'' is a graph by identifying u of G_1 and v of G_2 and adding a pendent edge uw, then

$$\alpha(G') \leq \alpha(G'').$$

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2.Graph Transformation

Theorem 7

(Guo 2010) Let G be a connected graph with at least two vertices. Let $G_{k,l}$ be a graph from G by attaching two paths of lengths k, l respectively, at vertex u of G; and let $G_{k+1,l-1}$ be a graph from G by attaching two paths of lengths $k+1, l-1$ respectively, at vertex u of G. If $k \geq l \geq 1$, then

$$\alpha(G_{k,l}) \geq \alpha(G_{k+1,l-1}).$$

2.Graph Transformation

Theorem 8

(Shao, Guo, Shan 2008)Let $vv_1, \cdots, vv_p$ *be pendant edges of a connected graph* G *on* n *vertices. Let* G' *be a graph from* G *by adding any* $0 \leq t \leq \frac{p(p-1)}{2}$ *edges among* $v_1, \cdots, v_p$*. If* $\alpha(G) \neq 1$*, then*

$$\alpha(G) = \alpha(G').$$

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2.Graph Transformation

Theorem 9

(Kirkland, Oliveira and Justel 2011) Let G be a graph on vertices $1, \cdots, n$, and suppose that vertex 1 of G has degree d. Select $p-1 \geq 1$ vertices of G, say $u_1, \cdots, u_{p-1}$ none of which is adjacent to vertex 1 in G. Let H be the graph on vertices $1, \cdots, n$ whose only edges are those between vertex 1 and each of vertices $u_1, \cdots, u_{p-1}$. If $G \bigcup H \neq K_n$, then $\alpha(G \bigcup H) - \alpha(G) \leq p - \varepsilon_0$, where ε_0 is the smallest positive root of the polynomial

$$d\varepsilon(p-\varepsilon) - (1-\varepsilon)^2(p-1-\varepsilon)^2.$$

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Bounds for the algebraic connectivity

Theorem 10

(Fiedler 1973) Let G be a connected graph of order n. Then

$$2(1-2\cos\frac{\pi}{n}) \leq \alpha(G) \leq n$$

with the left equality if and only if G is a path,the right equality if and only if $G = K_n$

Theorem 11

(Kirkland, Molitierno, Neumann and Shader 2002) Let G be a connected graph of order n with vertex connectivity $\nu(G)$. Then $\alpha(G) \leq \nu(G)$ with equality if and only if $G = G_1 \bigvee G_2$, where G_1 is a disconnected graph of order $n-\nu(G)$ and G_2 is a graph of order $\nu(G)$ with $\alpha(G_2) \geq 2\nu(G) - n$.

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Bounds for the algebraic connectivity

Theorem 12

(Belhaiza, Abreu, Hansen and Oliveira 2005) Let G be a simple graph of order n and size m. If $G \neq K_n$, then

$$\alpha(G) \leq \lfloor -1 + 2\sqrt{1+2m} \rfloor.$$

Theorem 13

(Mohar 1992) Let G be a graph of order n with diameter $diam(G)$. Then

$$\alpha(G) \geq \frac{4}{n diam(G)}.$$

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Bounds for the algebraic connectivity

Theorem 14

(Grone, Merris and Sunder 1990) Let T be a tree with diameter $diam(T)$. Then

$$\alpha(T) \leq 2(1 - \cos \frac{\pi}{diam(T)+1}).$$

Theorem 15

(Molitierno 2006) Let T be a planar graph. Then

$$\alpha(G) \leq 4$$

with equality if and only if $G = K_4$ or $G = K_{2,2,2}$.

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The dominating number

The dominating number $\gamma(G)$: The smallest number of $|S|$ such that for each vertex in $G - S$ is adjacent to one vertex in $S \subseteq V$.

Theorem 16

(Lu, Liu and Tian 2005) Let G be a connected graph with the dominating number $\gamma(G)$. Then

$$\alpha(G) \leq \frac{n(n - 2\gamma(G) + 1)}{n - \gamma(G)}$$

with equality if and only if $G = K_{2,2}$.

Theorem 17

(Nikiforov 2007) Let G be a connected graph of order n with the dominating number $\gamma(G) > 1$. Then $\alpha(G) \leq n - \gamma(G)$.

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The dominating number

Theorem 18

(Aouchiche, Hansen and Stevanovic 2010) Let G be a connected graph of order n with the dominating number $\gamma(G) \geq 3$.
(1) If $n = 2k \geq 6$, then $\alpha(G) \leq 2k - 2\gamma(G) + \frac{k+2-\sqrt{k^2+4}}{2}$.
(2) If $n = 2k+1 \geq 9$ with the minimum degree $\delta(G) \in \{1, 3, 5\}$ or δ is even and $G \notin \{F_6, F_7, F_8\}$, then
$\alpha(G) \leq 2k - 2\gamma(G) + \frac{k+3-\sqrt{(k+1)^2+4}}{2}$.

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The dominating number

Conjecture 19

(Aouchiche, Hansen and Stevanovic 2010) Let G be a connected graph of order $n = 2k + 1$ with the dominating number $\gamma(G) \geq 3$. If $G \notin \{A_3, A_4, F_6, F_7, F_8\}$, then

$$\alpha(G) \leq 2k - 2\gamma(G) + \frac{k + 3 - \sqrt{(k+1)^2 + 4}}{2}.$$

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The cut-vertex

Theorem 20

(Kirkland 2000, 2001) Let G be a connected graph of order n with k cut-vertices.

(1) If $2 \le k \le \frac{n}{2}$, then $\alpha(G) \le \frac{2(n-k)}{n-k+2+\sqrt{(n-k)^2+4}}$. with equality if and only if G is obtained from K_{n-k} by attaching a pendant vertex at each vertex in k vertices of K_{n-k}.

(2) If $k > \frac{n}{2}$ and there exist positive integer q and nonnegative integer l such that $k = \frac{qn+l}{q+1}$. Then $\alpha(G) \le \alpha(E_l(q,m))$, where $E_l(q,m)$ is defined as follows: starting with a graph H with m vertices which has at least r vertices of degree $m-1$ for $m \ge r \ge l$; select r such vertices at each attached a path of $q+1$ vertices, at each remaining vertex i of H attaching a path of j_i vertices subject to the condition $r + \sum_{i=1}^{m-r}(j_i - q) = l$.

Tree

Theorem 21

(Z 2004) Let T be a tree on n vertices with independence number β.
(i) If $\beta = n-1$, then $\alpha(T) \leq 1$ with equality if and only if T is $T_{n,n-1}$, i.e., T is the star $K_{1,n-1}$.
(ii) If $\beta = n-2$, then $\alpha(T) \leq \alpha(T_{n,n-2})$, where $\alpha(T_{n,n-2})$ is the smallest root of the following equation $\lambda^3-(n+2)\lambda^2+(3n-2)\lambda-n=0$. Moreover, equality holds if and only if T is $T_{n,n-2}$.
(iii) If $\beta < n-2$, then $\alpha(T) \leq \frac{3-\sqrt{5}}{2}$ with equality if and only if T is $T_{n,\beta}$.

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Extremal Graphs with Algebraic Connectivity

- Extremal graph theory is a branch of graph theory. One is interested in relations between the various graph invariants, such as order, size, connectivity, chromatic number, diameter and eigenvalues, and also in the values of these invariants which ensure that the graph has certain properties.
- Given a property $\mathfrak{P}$ and an invariant ψ for a class $\mathcal{H}$ of graphs, how to determine the smallest value m for which every graph G in $\mathcal{H}$ with $\psi(G) > m$ has property $\mathfrak{P}$ and with $\psi(G) = m$ are called the extremal graphs for the problem.
- In other words, for given an invariant ψ for a class $\mathcal{H}$ of graphs, determine all graphs with the maximum (minimum) values in $\mathcal{H}$.

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Extremal Graphs with Algebraic Connectivity

Theorem 22

(Fallat and Kirkland 1998) Let $\mathcal{T}_{n,d}$ be the set of all trees of order n with diamter d. The extremal trees that has the minimum (maximum) algebraic connectivity in $\mathcal{T}_{n,d}$ are unique. Moreover, this tree is obtained by identifying one pendant vertex of a path P_{d-1} of order $d-1$ and the center of $K_{1,\lfloor\frac{n-d+1}{2}\rfloor}$, and the other pendant vertex of P_{d-1} and the center of $K_{1,\lceil\frac{n-d+1}{2}\rceil}$ (by identifying one pendant vertex of a path P_{d-1} of order $d-1$ and the center of $K_{1,n-d+1}$).

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Extremal Graphs with Algebraic Connectivity

- The girth of a graph G is the length (number of vertices, or edges) of the shortest cycle in G.
- Let $\mathcal{H}_{n,g}$ be the set of all connected graphs of order n and girth g.
- How to determine extremal graphs with the maximum (minimum) algebraic connectivity in $\mathcal{H}_{n,g}$.

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Extremal Graphs with Algebraic Connectivity

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- Let $\mathcal{H}_{n,g}$ be the set of all connected graphs of order n and girth g.
- How to determine extremal graphs with the maximum (minimum) algebraic connectivity in $\mathcal{H}_{n,g}$.

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Extremal Graphs with Algebraic Connectivity

Conjecture 23

(Fallat and Kirkland 1998) The extremal graphs with the minimum algebraic connectivity in $\mathcal{H}_{n,g}$ are only lollipop graph $G_{n,g}$ that is obtained from g-cycle with a path of length $n-g$ joined at exactly one vertex on the cycle.

They prove a part result of this conjecture.

Theorem 24

(Fallat and Kirkland 1998) The extremal graphs with the minimum algebraic connectivity in $\mathcal{H}_{n,3}$ are only graph G that is obtained from 3-cycle with a path of length $n-3$ joined at exactly one vertex on the cycle.

Extremal Graphs with Algebraic Connectivity

Guo 2008 proves this conjecture.

Theorem 25

(Guo 2008) The extremal graphs with the minimum algebraic connectivity in $\mathcal{H}_{n,g}$ are only graph G that is obtained from g-cycle with a path of length $n-g$ joined at exactly one vertex on the cycle.

Fallat and Kirkland (1998) pointed out that determination of the graph on n vertices with fixed girth g that maximizes the algebraic connectivity appears to be more difficult.

Now there are part results.

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Extremal Graphs with Algebraic Connectivity

Let $\mathcal{U}_{n,g}$ be the set of all unicyclic graphs of order n and girth g.

Theorem 26

(Fallat and Kirkland 1998) The extremal graphs with the maximum algebraic connectivity in $\mathcal{U}_{n,3}$ are the graph $G_{n,3}$ that is obtained by taking a 3-cycle and appending $n-3$ pendant vertices to a single vertex on the cycle.

Theorem 27

(Fallat, Kirkland and Pati 2003)Fixed a girth g, there exists an N such that if $n > N$, then the extremal graphs with the maximum algebraic connectivity in $\mathcal{U}_{n,g}$ is $G_{n,g}$. In particular, for $g = 4$, the conjecture holds.

Extremal Graphs with Algebraic Connectivity

- Let $\mathcal{M}_{n,m}$ be all connected graphs of given order n and size m.
- (Biyikoglu and Leydold 2012) How to determine the extremal graphs with the maximum (minimum) algebraic connectivity in $\mathcal{M}_{n,m}$?
- The problem seems to be more difficult.
- For $n-1 \leq m \leq \frac{n(n-1)}{2} - 2$, there exists a $1 \leq t \leq n-2$ such that

$$\frac{(n-t)(n-t-1)}{2} + t \leq \frac{(n-t)(n-t-1)}{2} + n - 2.$$

- Then $m = \frac{(n-t)(n-t-1)}{2} + t + p$, where $1 \leq t \leq n-2$, $1 \leq p \leq n-t-1$.

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- Then $m = \frac{(n-t)(n-t-1)}{2} + t + p$, where $1 \le t \le n-2$, $1 \le p \le n-t-1$.

Extremal Graphs with Algebraic Connectivity

- A graph of order n with size m such that

$$\frac{(n-t)(n-t-1)}{2} + t \le m \le \frac{(n-t)(n-t-1)}{2} + n - 2$$

is called (n, p, t) path-complete graph, denoted $PC_{n,p,t}$ if and only if

(1) the maximal clique of $PC_{n,p,t}$ is K_{n-t}.

(2) has a path of order $P_{t+1} = \{v_0, v_1, v_2, \cdots, v_t\}$ such that $v_0 \in K_{n-t} \bigcap P_{t+1}$ and v_1 is joined to K_{n-t} by p edges;

(3) there are no other edges.

Extremal Graphs with Algebraic Connectivity

Conjecture 28

(Belhaiza, Abreu, Hansen and Oliveira 2005) The extremal graphs with the minimum algebraic connectivity in $\mathcal{M}_{n,m}$ for $n-1 \le m \le \frac{n(n-1)}{2} - 1$ are all path-complete graphs.

Conjecture 29

(Belhaiza, Abreu, Hansen and Oliveira 2005) For each $n > 3$, the minimum algebraic connectivity of a graph G with n vertices and m edges is an increasing, piecewise concave function of m. Moreover, each concave piece corresponds to a family of path-complete graphs. Finally, for $t = 1$, $\alpha(G) = \delta(G)$, and for $t > 2, \alpha(G) \le 1$.

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Extremal Graphs with Algebraic Connectivity

Theorem 30

(Belhaiza, Abreu, Hansen and Oliveira 2005) For all $\frac{(n-1)(n-2)}{2} \leq m \leq \frac{n(n-1)}{2}$, the extremal graphs G with the maximum algebraic connectivity in $\mathcal{M}_{n,m}$ has the property that the complement of G is the disjoint union of triangles K_3, paths P_3, edges K_2 and isolated vertices K_1.

- Biyikoglu and Leydold (2013)investigate the structure of connected graphs of given size and order that have minimal algebraic connectivity.
- How about other value of size?

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Extremal Graphs with Algebraic Connectivity

- In 1941,Turán determined the maximal number of edges of a graph G which does not contain a copy of the complete graph K_{r+1}, which started the research of the extremal theory of graphs.
- Let $T_{n,r}$, called *Turán graph*, be the complete r-partite graph of order n, and the size of every class of which is $\lceil \frac{n}{r} \rceil$ or $\lfloor \frac{n}{r} \rfloor$.

Theorem 31

(Turan 1941) Let G be a graph of order n not containing K_{r+1}. Then $e(G) \leq e(T_{n,r})$ with equality holding if and only if $G = T_{n,r}$, where $e(G)$ is the number of edges in G.

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Extremal Graphs with Algebraic Connectivity

- Erdős and Stone (1946), and Erdős and Simonovits (1966) expanded the above results.
- Let $\mathcal{H}$ be the set of graphs and $\chi(H)$ be the chromatic number of H, and let $\psi(\mathcal{H}) = min\{\chi(H)|H \in \mathcal{H}\} - 1$

Theorem 32

(Erdős-Stone-Simonovits theorem) Let $ex(n, \mathcal{H})$ be the maximum number of edges of a graph with order n not containing a copy of any graph in $\mathcal{H}$. If $\psi(\mathcal{H}) > 1$, then

$$\lim_{n\to\infty} \frac{ex(n,\mathcal{H})}{\binom{n}{2}} = 1 - \frac{1}{\psi(\mathcal{H})}.$$

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Extremal Graphs with Algebraic Connectivity

- Are there similar results for the algebraic connectivity?
- Yes. There is an analogy for Erdős-Stone-Simonovits theorem in spectral graph theory.
- Characterize all graphs of order n not containing a complete subgraph K_r which have the maximum and minimum algebraic connectivity.

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Extremal Graphs with Algebraic Connectivity

Theorem 33

(Jin and Z 2013) Let $\alpha(n, \mathcal{H})$ *be the largest algebraic connectivity of graphs of order* n *without containing a copy of any graph* H *in* $\mathcal{H}$. *Then*

$$\lim_{n\to\infty} \frac{\alpha(n, \mathcal{H})}{n} = 1 - \frac{1}{\psi(\mathcal{H})},$$

where $\psi(\mathcal{H}) = min\{ \ \chi(H) | \ \ H \in \mathcal{H} \ \} - 1.$

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Extremal Graphs with Algebraic Connectivity

Theorem 34

(Jin znad Z 2013) Let G be a non-complete graph of order n not containing K_{r+1}. Then

$$\alpha(G) \leq n - \lceil \frac{n}{r} \rceil = \alpha(T_{n,r}), \tag{1}$$

where $\lceil a \rceil$ is the least integer no less than a. Moreover, if $n = kr$ or $n = kr + r - 1$, then equality (1) holds if and only if G is Turán graph $T_{n,r}$. If $n = kr + t, 0 < t < r - 1$, then equality (1) holds if and only if there exist graphs $H_1, \ldots, H_t$ of order $k + 1$ with no edges and H of order $n - (k+1)t$ not containing K_{r+1-t} such that $G = H_1 \vee H_2 \cdots \vee H_t \vee H$ and $\alpha(H) \geq n - (k+1)(t+1)$.

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Extremal Graphs with Algebraic Connectivity

Theorem 35

(Jin and Z 2013) Let G be a connected graph with the clique number $r \geq 2$. Then

$$\alpha(G) \geq \alpha(Ki_{n,r}), \tag{2}$$

where $Ki_{n,r}$ is a kite graph of order n which is obtained by adding a pendant path of length $n-r$ to a vertex of K_r. Moreover, equality (2) holds if and only if $G = Ki_{n,r}$.

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Random graphs

- G be an ER random graph: For labeling n vertices $\{v_1, \cdots, v_n\}$, the probability that two vertices v_i and v_j are adjacent is p and each edge is independent.

Theorem 36

(Juhasz 1991) Let G be an ER random graph of order n with the probability p. For any $\varepsilon > 0$, we have

$$\alpha(G) = pn + o(n^{1/2+\varepsilon}), \text{ in probability.}$$

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Random graphs

Theorem 37

(Gu, Z and Zhou 2010) Let $\mathcal{S}(n, c, k)$ be the small-world network with n nodes, which is a union of an Erdös-Réyni random graph $\mathcal{G}(n, \frac{c}{n})$ and a $2k$ regular cycle. Then the algebraic connectivity of $\mathcal{S}(n, c, k)$ is almost surely bounded below by

$$\frac{k^2c^2 \log\log n}{2(k+1)^2 \log^3 n}. \tag{3}$$

Random graphs

Olfati and Saber (2008) defined

$$\gamma_2(n,c,k) = \frac{\lambda_2(n,c,k)}{\lambda_2(n,0,k)} \tag{4}$$

to be the *algebraic connectivity gain* of $\mathcal{S}(n,c,k)$.

Theorem 38

(Gu, Z and Zhou 2010) The algebraic connectivity gain of the small-world network $\mathcal{S}(n,c,k)$ follows almost surely inequality

$$\gamma_2(\mathcal{S}(n,c,k)) \geqslant \frac{3kc^2n^2 \log\log n}{2(k+1)^3(2k+1)\pi^2 \log^3 n}. \tag{5}$$

Random graphs

- The above results give a mathematical rigorous estimation of the lower bound for the algebraic connectivity of the small-world networks, which is much larger than the algebraic connectivity of the regular circle.
- This result explains why the consensus problems on the small-world network have a ultrafast convergence rate and how much it can be improved.
- It also characterizes quantitatively what kind of the small-world networks can be synchronized.

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Thank you very much for attention!